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Sheaf Homology and Complete Reducibility

STEPHEN D. SMITH*

*Department of Mathematics, Statistics and Computer Science,
University of Illinois at Chicago,
Chicago, Ilinois 60680*

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SUMMARY

In the study of finite simple groups by means of the geometries provided by their local subgroups, problems of structure often reduce to questions about modular representations in finite characteristic. Of particular interest are modules spanned by fixed points of Sylow groups (including "high weight" modules for Chevalley groups), for which the homology methods of Ronan and Smith [6] can be very useful. This note presents an elementary sufficient condition for splitting of some reducible modules of this type. Among the applications, we generalize a number of results appearing in recent work of Timmesfeld [11], which provided the original inspiration for this analysis. Some of the ideas had appeared, in a special case, in [10].

SHEAVES AND SPLITTING OF EXTENSIONS

We will restrict attention to the case of Chevalley groups; with due care, it is possible to develop analogues for the twisted groups, and for sporadic-group geometries as in Ronan and Smith [5] or Ronan and Stroth [7] (details will appear in [12]).

Henceforth let G denote a (universal) Chevalley group, with simple system Π of rank n, defined for the finite field $\mathbb{F}_q$ of characteristic p. As our point of view is geometric, we assume unless otherwise specified that rank $n \geqslant 2$; and so must also assume results [1] about the representations of the rank-one group $\mathrm{SL}_2(q)$. We fix a Borel subgroup B of G; conjugacy classes of parabolic subgroups may be represented by those containing B, which are in turn determined by subsets $J \subseteq \Pi$. Since maximal parabolics are

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basic in the zeroth homology construction of [6], it is convenient to adopt the opposite of the usual convention, so that the parabolic subgroup P_J (with unipotent radical U_J) has Levi complement L_J, a reductive group with simple system $\Pi - J$. In particular, our Borel subgroup B is just P_Π; we abbreviate its unipotent radical U_Π by U, which is a Sylow p-group of G; a Levi complement L_Π is then a Cartan subgroup, and may be denoted by the customary H.

Our study will be focused on modules satisfying the basic "high weight" condition:

HYPOTHESIS A. V is an $\mathbb{F}_q G$-module satisfying $V = \langle C_V(U)^g : g \in G \rangle$.

Since U is a p-group for $p = \operatorname{char}(\mathbb{F}_q)$, we have $C_V(U) \neq 0$, so the right side is always a non-zero submodule; but in general it can be proper. Examples of Hypothesis A are given by irreducible modules, "Weyl modules," induced modules $1_{P_J}^G$, and so on. Clearly the condition is inherited by quotient modules; an easy example of its failure to carry over to submodules is provided by $G = \mathrm{Sp}_4(2)$, $V = 1_{P_1}^G$, where $C_V(U)$ meets just four of the six composition factors.

Since we are specifically concerned with splitting of extensions, we will assume further that we have reduced to the minimal situation:

HYPOTHESIS B. V has submodule S and quotient $V/S \cong T$, where S and T are irreducible.

Of course, if $V \cong S \oplus T$ then Hypotheses A and B hold automatically, but not conversely, as can be seen in the example $G = \mathrm{Sp}_4(2)$, acting as S_6 on $V = \{$the even subsets of a 6-set$\}$: V is a 5-dimensional indecomposable with submodule S trivial, and top quotient T a natural symplectic space. Nonetheless, experience suggests that "typical" indecomposables satisfying Hypothesis B have $C_V(U) \leqslant S$, and so fail Hypothesis A. For example, if T is trivial, then Hypothesis A implies splitting of V by the standard criterion of Gaschütz. So we may seek to obtain precise conditions for Hypothesis A to hold with Hypothesis B. Our tool will be the study of sheaves and zeroth homology developed in [6], which we describe only briefly here.

To define a "sheaf" on V (see [6, Sect. 1 and remarks on (2.2)]) it suffices (by flag-transitive action of G on its building) to give just the terms for the parabolics P_J above our fixed Borel subgroup B. So for this paper, a *sheaf on* V will mean an assignement to each (proper) P_J of some P_J-invariant subspace V_J of $C_V(U_J)$; with an inclusion $V_J \leqslant V_K$ whenever $P_J \leqslant P_K$ (that is, when $J \supseteq K$). In practice we shall mostly use the full "fixed-point" sheaf $\mathscr{F}_V$ of [6], so that the notation V_J will mean the full centralizer $C_V(U_J)$, unless otherwise specified.

The easy sheaf-theoretic consequence of Hypotheses A and B is:

(C) $\mathscr{F}_V$ has sub-sheaf $\mathscr{F}_S$, with quotient sheaf $\mathscr{F}_T$.

Proof. Since S is a submodule, the sheaf inclusion $\mathscr{F}_S \subseteq \mathscr{F}_V$ is automatic. Now let bars denote images in the quotient V/S, but identify the full quotient $\bar{V}$ with T. Since T is irreducible, T_Π (that is, $C_T(U)$) must be 1-dimensional by Steinberg's well-known theorem. But $0 \neq \overline{V_\Pi}$ by Hypothesis A, and $\overline{V_\Pi} \leqslant T_\Pi \equiv C_{\bar{V}}(U)$ is immediate, so we conclude $\overline{V_\Pi} = T_\Pi$. Now it is a consequence of the "irreducible/parabolic" result of [9] that for each J, T_J is generated by L_J-conjugates of T_Π; so from the previous remark (and the obvious inclusions $V_\Pi \leqslant V_J$ and $\overline{V_J} \leqslant T_J$) we get $\overline{V_J} = T_J$. By the usual isomorphism theorems, we deduce $T_J = \overline{V_J} = (V_J + S)/S \cong V_J/(V_J \cap S) = V_J/S_J$. So we see that the quotient of $\mathscr{F}_V$ by the sub-sheaf $\mathscr{F}_S$ is indeed $\mathscr{F}_T$. ∎

Just as we can ask if the module V in Hypothesis B splits, we can ask if the sheaf $\mathscr{F}_V$ in (C) splits. Trivially if $V = S \oplus T$ we get $\mathscr{F}_V = \mathscr{F}_S \oplus \mathscr{F}_T$ correspondingly; but the converse fails, as can be checked in the 5-dimensional example for $\mathrm{Sp}_4(2)$ mentioned above. However, when the sheaf $\mathscr{F}_V$ does split as $\mathscr{F}_S \oplus \mathscr{F}_T$, but V remains indecomposable, the spaces in the sub-sheaf $\mathscr{F}_T$ must generate all of V. Then by [6, (1.2)] V must be a quotient of the homology module $H_0(\mathscr{F}_V)$. This remark establishes our main technical result:

PROPOSITION 1. *Assume Hypotheses A and B with*:

(i) $\mathscr{F}_V = \mathscr{F}_S \oplus \mathscr{F}_T$.

(ii) $H_0(\mathscr{F}_T)$ *has no quotient satisfying Hypothesis B.*

Then $V = S \oplus T$ *under action of* G.

Thus the effect of the splitting of the sheaf in (i) is to transfer the problem of an abstract extension V to consideration of quotients of the explicitly constructible module $H_0(\mathscr{F}_T)$.

We remark first that there may be some subtlety in establishing Proposition 1(i). Consider, for example, $G = L_3(2)$ and V a 6-dimensional permutation module obtained from an inclusion $L_3(2) < A_7$ (equivalently, from $1_{P_1}^G$). Here we find that each subspace V_J splits naturally as $S_J \oplus T_J$ under action of L_J; but the spaces T_J do not admit the connecting maps required for a subsheaf $\mathscr{F}_T$. (If V has permutation-module basis $\{a, b, c, d, e, f, g\}$, a suitable choice of P_1 and P_2 gives $T_1 = \langle (abcdef) \rangle$, not contained in $T_2 = \langle (ef), (gh) \rangle$.)

Secondly, we note as in [6, Sect. 3] that there is a straightforward if tedious algorithm for computing $H_0(\mathscr{F}_T)$, given fixed G and T, and so for checking Proposition 1(ii). One frequently useful shortcut is mentioned in [6, end of Section 3]; it suffices to find some J for which the induced module T_J^G has no quotient satisfying Hypothesis B. Alternatively, we can sometimes show that $H_0(\mathscr{F}_T)$ is just T itself, and so has no quotient satisfying Hypothesis B.

In inductive arguments, we will need to consider minimal parabolics, with Levi complements of rank 1. Now for $\mathrm{SL}_2(q)$, the only proper parabolics are Borel subgroups B, and the analogue of $H_0(\mathscr{F}_V)$ is just the induced module V_Π^G. For this situation, we require the following (fairly well-known) statement:

(D) Let $G \cong \mathrm{SL}_2(q)$ for $p \neq 3$, and let V be a module satisfying Hypothesis A, all of whose composition factors are 2-dimensional natural modules.

Then V is completely reducible.

Proof. See Corollary 4.5 of [1] (a more general result). ∎

Note that in the one exceptional case $p = 3$, our later inductive arguments must avoid appealing to (D). (This necessitates some extra work.)

Applications

We turn now to some specific situations in which the sufficient conditions of Proposition 1 can be verified. Our initial analysis wil be for the case where the composition factors S and T are the same. We require a standard fact (see [6], proof of (2.5)), relevant to Proposition 1(i).

LEMMA 2. *Let $X \cong Y \oplus Y$ for irreducible $\mathbb{F}_q G$-module Y. Then X contains just $q+1$ submodules $\cong Y$. In particular, a proper sub-sheaf $\mathscr{F}_Y$ of $\mathscr{F}_X$ generates just a single submodule $\cong Y$.*

Notation. As in [6, Sect. 3] it will be convenient to abbreviate the homology module $H_0(\mathscr{F}_T)$(for some module T) by $\hat{T}$.

In order to work inductively, we consider the class of "minimal" modules: since if T is the irreducible for a high weight λ which is minimal (see Humphreys [4, p. 72]), then we easily check that the minimality condition is inherited by each T_J considered as L_J-module. (In particular, T_J is non-trivial for at most one quasi-simple component of the reductive group L_J). This will be useful because $\hat{T}$ has been computed for these cases:

THEOREM 3 [6, (4.1)]. *For T minimal, either $\hat{T} \cong T$, or $G \cong \mathrm{Sp}_{2n}(2^m)$ and $\hat{T}$ is the extension of a trivial module by a natural (symplectic) module.*

Now it is a matter of simple induction to obtain:

PROPOSITION 4. *Let V be a module with Hypothesis A, such that all composition factors are isomorphic to a fixed minimal module T. Then V is completely reducible.*

Proof. We assume $p \neq 3$ as the "generic" argument invokes (D); details for $p = 3$ appear in the Appendix. By induction on the number of composition factors, we may assume that Hypothesis B holds with $S \cong V/S \cong T$. We begin by checking that each V_J splits under L_J, temporarily postponing the question of connecting maps for a sheaf. When $1 \leqslant |J| \leqslant |\Pi| - 2$, we get $V_J \cong T_J \oplus T_J$ by induction on $|\Pi|$: for then L_J has rank $\geqslant 2$, T_J is a minimal module for L_J, $S_J \cong V_J/S_J \cong T_J$ by (C). For $|J| = |\Pi| - 1$, the module T_J can be only a trivial or natural module for $SL_2(q)$, (since minimal λ is always fundamental). For natural T_J, we get splitting by (D). For T_J trivial, we see V_J must be trivial under at least the commutator L'_J of L_J, hence split under all of L_J except possibly in case $q = 2$ when $L_J \cong SL_2(q)$ is solvable. (The solvable case $q = 3$ is excluded by hypothesis.) For $q = 2$, note $V_J = V_\Pi$ is also trivial under a full unipotent group U_Π, which complements $SL_2(q)'$ in $SL_2(q)$. Finally, for $J = \Pi$, splitting follows from the completely reducible action of $L_\Pi = H$.

Now from Lemma 2 we see we have for each J a choice of complement to S_J in V_J. We wish to make choices so that inclusion maps define a natural sub-sheaf complementing $\mathscr{F}_S$. We begin by choosing one of the complements to S_Π in V_Π under action of $L_\Pi = H$, and denote it by T_Π. By Lemma 2, conjugates of T_Π under L_J generate a single submodule T_J of V_J, a complement to S_J by irreducibility. This generation from T_Π defines connecting maps, so that the spaces T_J afford a subsheaf $\mathscr{F}_T$ as in Proposition 1(i). But Proposition 1(ii) holds by Theorem 3, so Proposition 4 is proved. ∎

This result "essentially" generalizes (2.3) and (2.8) of Timmesfeld [11]. Actually Timmesfeld works $\mathbb{F}_q$-modules read over the prime field $\mathbb{F}_2$, whereas for simplicity we have considered modules only over the natural splitting field for G. Techniques for Timmesfeld's situation (and the more complicated analogue of Proposition 4) are described in the latter part of Section 2 of [6].

We may ask whether the "minimal" hypothesis in Proposition 4 is actually necessary, for we see the proof involves only induction and the analogous result (D) for rank 1. The nuisance for $p = 3$ generalizes: for any odd p, if λ is the involution-character of $H \cong P_\Pi/U_\Pi$ in $G = SL_2(p)$, then the induced module affording λ^G is indecomposable with two identical composition factors. Experience seems to indicate that this does not propagate to higher rank; we ask:

QUESTION 5. Suppose V satisfies Hypothesis A with all composition factors isomorphic. Must V be completely reducible?[1]

Unfortunately, a proof could not proceed by trying to show simply that $H_0(\mathscr{F}_T)$ involves T only once as a composition factor. For in [6, Example 5] we see that for $G = G_2(2)$ and V of dimension 6, that $H_0(\mathscr{F}_V)$ involves V twice (though not in any quotient satisfying Hypothesis B).

We turn next to some cases with $S \not\cong T$. We can get further mileage out of Theorem 3 when both S and T are minimal modules: in such cases (for S non-trivial) we need only establish Proposition 1(i).

Some immediate obstacles to extending Proposition 4 for differing minimal composition factors are provided by the non-split extension of a trivial module by a natural module for $\mathrm{SL}_2(q)$ $(q > 2)$, and in rank 2 by our earlier 6-dimensional example for $G = L_3(2)$. Let us pursue this latter example into higher rank: For $G = \mathrm{SL}_{n+1}(2)$, we find that the induced module $1_{P_1}^G$ is the sum of a trivial module and a uniserial module whose (descending) factors are the minimal modules for weights $\lambda_1, \lambda_2, \ldots, \lambda_n$. Any consecutive pair provides a section satisfying Hypotheses A and B but not Proposition 1(i). If we try to show all examples are of this type, we stumble on one more: For G of type BC_3 in characteristic 2, the Weyl module of type C_3 for weight λ_3 is a 14-dimensional indecomposable, with 6-dimensional symplectic submodule for weight λ_1 (minimal for type C_3), and 8-dimensional spin-module quotient (weight λ_3 is minimal for type B_3). With these in hand, we state:

PROPOSITION 6. *Suppose V satisfies Hypothesis A, with all composition factors non-trivial minimal modules for various λ_i. Assume no pair $\{i, i'\}$ is adjacent in the Dynkin diagram. (If $p = 2$, assume also G is not of type BC_3.) Then V is completely reducible.*

Remark. Note "non-adjacency" of distinct minimal weights is automatic outside type A_n and BC_2 (for $p = 2$). The result is then covered by Proposition 4 unless there are distinct i, i': so the non-adjacency hypothesis effectively gives rank $n \geqslant 3$.

Proof. Using induction on the number of composition factors, we may assume V satisfies Hypothesis B with modules S, T for weights λ_s, λ_t; and $s \neq t$ by Proposition 4. Note this means that S_Π and T_Π are 1-spaces invariant under P_s and P_t, respectively. Now V_Π is 2-dimensional, and among its $q+1$ B-invariant 1-subspaces are S_Π, and another we *denote* by T_Π, distinguished by having full stabilizers (in G) equal to P_s nd P_t, respectively; the other $(q-1)$ B-invariant 1-spaces are stabilized only by P_{st}. For each J we set $T_J \equiv \langle T_\Pi^{L_J} \rangle$, and we wish to show each such T_J is a

[1] An example found by Jantzen answers this question in the negative.

complement to S_J in V_J (instead of covering all of V_J). Since $P_J = \bigcap_{k \in J} P_k$ and $V_J = \bigcap_{k \in J} V_k$, it will suffice to consider maximal P_k.

So consider $k \neq s, t$. Assume first that s and t lie in the same connected component of the Dynkin diagram for $\Pi - \{k\}$; then our non-adjacency hypotheses are inherited by L_k (the component is not BC_3, since s and t must be end nodes in type BC_n). Thus by induction on $|\Pi|$, V_k splits under L_k. Then P_{tk} normalizes a 1-space in a complement to S_k; as $P_{tk} \nleq P_{st}$, this 1-space can only be T_Π, and so the complement is our T_k.

Assume instead that t lies in one connected component J of $\Pi - \{k\}$, and s in the union I of the other components. We may choose complements so that L_k is the central product of normal subgroups L_{kI} and L_{kJ}. Now L_{kJ} normalizes T_Π, so $T_k \equiv \langle T_\Pi^{L_k} \rangle = \langle T_\Pi^{L_{kJ} L_{kI}} \rangle = \langle T_\Pi^{L_{kI}} \rangle \equiv T_{kI}$. Further, T_Π is centralized by $U_{kI} \leqslant U_k L_{kJ}$, which is normalized by L_{kI}, so that all of T_k is centralized by U_{kI}. Now U_{kI} covers a Sylow group of L_{kJ}, so that by Gaschütz's standard criterion, V_k splits under action of L_{kJ}. A complement to S_k is then determined as $C_{V_k}(L'_{kJ})$, which is P_k-invariant by normality of L_{kJ} in L_k. Just as before, this complement contains T_Π and so must be our T_k.

This leaves cases $k = s$ or t. For P_t, we observe that T_Π is L_t-invariant, and so $T_t \equiv T_\Pi$ is the desired complement. So it remains to consider P_s. Note for each $k \neq s$ (including $k = t$) we got $V_k = S_k \oplus T_k$; it follows that $V_{sk} = S_{sk} \oplus T_{sk}$; and then by intersections that $V_{sJ} = S_{sJ} \oplus T_{sJ}$ for all $\varnothing \neq J \subseteq \Pi - \{s\}$, with appropriate connecting maps. Thus we have Proposition 1(i) for L_s on V_s. But we also get Proposition 1(ii) by Theorem 3 applied to L_s on T_s: since non-adjacency guarantees that the component of L_s acting non-trivially on V_s/S_s (from the connected component of t in $\Pi - \{s\}$) has rank $\geqslant 2$; and also by non-adjacency, T_s cannot be a natural symplectic module, except in case BC_3 $(p = 2)$, with $s = 1$ and T_1 the natural $(\overset{2}{\bullet}\!=\!\overset{1}{\bullet})$ module for $L_1 \cong \mathrm{Sp}_4(2)$, which is excluded by hypothesis. Thus V_s splits by Proposition 1. And as before a complement is generated by conjugates of T_Π and so its must be our T_s.

Now we have $V_J = S_J \oplus T_J$ for all J as remarked earlier, giving Proposition 1(i) for G; and Proposition 1(ii) is again provided by Theorem 3. So Proposition 6 follows. ∎

Proposition 6 may be compared to (2.5)(2) of Timmesfeld [11]. It also provides an easier proof of the splitting result in the author's [10]. Indeed the present proof is even characteristic free.

APPENDIX

We must show that the failure of (D) for $SL_2(3)$ does not invalidate Proposition 4. We simply replace (D) by the necessary rank-2 results, provided directly. (Recall G_2 has no non-trivial minimal module). The first of these was provided, for this purpose, by David Wales:

LEMMA 7. (Wales). *Assume G has type A_2 and $p = 3$. Assume V satisfies Hypotheses A and B with 3-dimensional composition factors. Then V is completely reducible.*

Proof. It will suffice to consider the case $q = p = 3$: For any larger group $G = SL_3(q)$ contains a subgroup $SL_3(3)$; if the result holds for this $SL_3(3)$, then a root group acts quadratically on V. Hence the same holds for a root group of G on V. But then for minimal parabolics P_i ($i = 1$ or 2) the module V_i is not the non-split extension violating (D). (See [3, Lemma 4.8], for example.) So the proof of Proposition 4 goes through, showing V is split.

Thus we may assume $G = SL_3(3)$. Then $G = \langle M, N \rangle$, where $M \cong S_4$ consists of monomial matrices of determinant 1, and N is a Sylow 2-group (of order 16):

$$\left\langle \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\rangle.$$

Now the action of N decomposes both constituents of V, so that we may write $V_N = V_2 \oplus V_4$, where N has 2-dimensional constituents on V_4, and acts by scalars on V_2. On the other hand, the 3-dimensional constituents S and V/S of V are both irreducible under M, hence projective since $|M| = 2^3 \cdot 3$, so that V_M is completely relducible. Now let T denote one of the three M-complements to S in V (cf. Lemma 2), chosen so that $T \cap V_4$ is an N-complement to $S \cap V_4$ in V (the three possible $T \cap V_4$ are invariant under $M \cap N \leqslant N$, and so are permuted by $N/(M \cap N)$ of order 2; hence at least one is fixed). Thus N preserves $T \cap V_4$, and also $T \cap V_2$ by scalar action; so N preserves $T \equiv (T \cap V_4) \oplus (T \cap V_2)$; and then all of $G = \langle M, N \rangle$ preserves T. ∎

For G of type $B_2 = C_2$, the only minimal module is the 4-dimensional spin (i.e., symplectic) module. We may begin as in the proof of Lemma 8, choosing as M the normalizer of an element of order 5, with cyclic Sylow 2-group E of order 8 (cf. Srinivasan [8, pp. 495–496]), and let N be a subgroup of order 16 of $N(E)$, inducing $x \to x^3$ on E. Then N stabilizes all the 2-dimensional E-invariant subspaces of $V - S$, including those three pairs affording M-complements to S. This shows splitting under $\langle M, N\rangle \cong GL_2(5)$. But then a 3-element of $\langle M, N\rangle$ acts quadratically and fixed-point-freely on V. This means that the "local subspace" V_J involving two 2-dimensional $\mathrm{SL}_2(3)$-modules cannot be the non-split module which is the obstruction to (D). Thus the proof of Proposition 4 goes through for this case.

We can now establish the general case of Proposition 4 when $p=3$. We may assume G has rank at least 3, since the rank-2 cases were just handled. Any minimal parabolic P_J lies in a rank-2 parabolic P_k with L'_k of type $\mathrm{SL}_3(q)$ or $\mathrm{Sp}_4(q)$. If S_J is the natural $\mathrm{SL}_2(q)$-module, then the minimal-module hypothesis guarantees that S_k is a 3- or 4-dimensional module just dealt with. Then splitting of $V_k = S_k \oplus T_k$ under L_k gives splitting of V_J under L_J. The remainder of the proof of Proposition 4 proceeds as in the general case. (When $q=3$, V_Π again complements $\mathrm{SL}_2(q)'$ in $\mathrm{SL}_2(q)$.)

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